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Non-linear wave interactions in a complex Hamiltonian formalism

Frank Verheest

Instituut voor theoretische mechanika, Rijksuniversiteit Gent, Krijgslaan 281 (S9), B-9000 Gent, Belgium

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Abstract. A Hamiltonian formalism is used to study the set of equations governing the changes in the amplitudes of some coherent waves due to their non-linear interactions. The main point of the present description is that it uses these complex wave amplitudes as canonical variables for a real Hamiltonian, thus greatly simplifying the algebra. Three examples are then dealt with. First, the Hamiltonian for the four-wave interactions is derived, in accordance with previous studies. Such systems are then known to be integrable. As a second example, one looks at a very special case of three interacting waves, where a fundamental interacts simultaneously with its second and third harmonic. Even though this system can be Hamiltonian, it seems to be non-integrable. Finally, attention is focused on the elliptically polarised third-harmonic generation, both in anisotropic and in isotropic media. This set of equations can be Hamiltonian under certain restrictions on the coupling coefficients, but it also appears to be non-integrable, because not enough independent invariants are found.

1. Introduction

Although the subject of non-linear interaction between some coherent waves has been dealt with rather extensively, both in plasma physics and in other fields (see, e.g., Weiland and Wilhelmsson (1977) for a rather comprehensive survey as far as plasma physics is concerned), attention has recently been focused on the Hamiltonian description of such phenomena. This is mainly due to the search for integrability (or non-integrability) among multiple three-wave systems (e.g. Meiss 1979), where each triplet of waves shares a common wave.

The three-wave case can easily be shown to be Hamiltonian and was already long known to be integrable, whereas for more complicated sets of interacting waves such properties are not always obvious or possible. It is the aim of this paper to look at three different applications of the Hamiltonian formalism to wave interactions in plasma physics: four-wave interactions, second- and third-harmonic generation and interaction between elliptically polarised waves. In these examples, full use will be made of the fact that in general the equations governing the wave amplitudes are written in complex notation and for such cases a Hamiltonian description in complex canonical variables was evolved elsewhere (Verheest 1985), where the complex wave amplitudes themselves serve as the canonical variables. As will become clear in the subsequent examples, this procedure greatly simplifies the algebra, compared to more traditional descriptions.

2. Hamiltonian description in complex variables

One is interested in a set of interacting waves, which in the linear approximation are just superposed:

$$u_{\text{lin}} = \sum_{j=1}^{N} a_j(t_{\text{slow}}) \exp \mathrm{i}(\boldsymbol{k}_j \cdot \boldsymbol{x} - \omega_j t_{\text{fast}}) + \text{cc}. \tag{2.1}$$

Due to the non-linear interactions, the waves will see their amplitudes a_j change slowly, according to

$$\dot{a}_j \equiv \frac{\mathrm{d}a_j}{\mathrm{d}t_{\text{slow}}} = \mathrm{i}f_j(a_1, \ldots, a_N, \bar{a}_1, \ldots, \bar{a}_N; t). \tag{2.2}$$

The usual procedure for wave-wave interactions in a Hamiltonian formulation (Falk 1982, Menyuk *et al* 1983) has been to introduce as canonical variables the actions J_j and the angles ϕ_j, such that

$$a_j = \sqrt{J_j} \exp \mathrm{i}\phi_j \tag{2.3}$$

with

$$\dot{J}_j = -\partial H/\partial \phi_j \qquad \dot{\phi}_j = \partial H/\partial J_j. \tag{2.4}$$

The Hamiltonian H is here a function of all J_j and ϕ_j (and possibly t). To check whether (2.2) is a Hamiltonian system, one splits it into its real and imaginary parts and uses the integrability conditions derived from (2.4):

$$\frac{\partial \dot{J}_j}{\partial \phi_l} = \frac{\partial \dot{J}_l}{\partial \phi_j} \qquad \frac{\partial \dot{\phi}_j}{\partial J_l} = \frac{\partial \dot{\phi}_l}{\partial J_j} \qquad \frac{\partial \dot{J}_j}{\partial J_l} = -\frac{\partial \dot{\phi}_l}{\partial \phi_j}. \tag{2.5}$$

In view of the fact that (2.2) is given in complex variables, it seems advantageous to retain these variables as canonical variables and derive from (2.3) and (2.4)

$$\dot{a}_j = \mathrm{i}\frac{\partial H}{\partial \bar{a}_j} \qquad \dot{\bar{a}}_j = -\mathrm{i}\frac{\partial H}{\partial a_j} \tag{2.6}$$

where H is now a function of all a_j and $\bar{a}_j$ instead of all J_j and ϕ_j. The integrability conditions now become

$$\frac{\partial \dot{a}_j}{\partial \bar{a}_l} = \frac{\partial \dot{a}_l}{\partial \bar{a}_j} \qquad \frac{\partial \dot{\bar{a}}_j}{\partial a_l} = \frac{\partial \dot{\bar{a}}_l}{\partial a_j} \qquad \frac{\partial \dot{a}_j}{\partial a_l} = -\frac{\partial \dot{\bar{a}}_l}{\partial \bar{a}_j}. \tag{2.7}$$

The last of these conditions can also be used in the form

$$\frac{\partial \dot{a}_j}{\partial a_l} = -\overline{\frac{\partial \dot{a}_l}{\partial a_j}}. \tag{2.8}$$

Applying this to (2.2), one need only check that

$$\frac{\partial f_j}{\partial \bar{a}_l} = \frac{\partial f_l}{\partial \bar{a}_j} \qquad \frac{\partial f_j}{\partial a_l} = \overline{\frac{\partial f_l}{\partial a_j}} \tag{2.9}$$

for the system to be Hamiltonian. The Hamiltonian itself can be found from

$$\partial H/\partial \bar{a}_j = f_j \tag{2.10}$$

with the proviso that it be a real function of its complex arguments.

In the following sections we will give some examples of how one can usefully apply the above ideas to several wave interactions cropping up in non-linear plasma physics and related areas of physics.

3. Four-wave interactions

When using symmetric selection rules for the wavenumbers and the frequencies,

$$\sum_{j=1}^{4} \boldsymbol{k}_j = \boldsymbol{0} \qquad \sum_{j=1}^{4} \omega_j = \delta \simeq 0 \tag{3.1}$$

including a small frequency mismatch, one can write the equations for the slow evolution of the amplitudes of four interacting waves (Verheest 1982a) as

$$\dot{a}_j = \frac{\mathrm{i}}{\bar{a}_j} \bar{a}_1 \bar{a}_2 \bar{a}_3 \bar{a}_4 \exp \mathrm{i}\delta t_{\mathrm{slow}} + \mathrm{i} \sum_{l=1}^{4} \mu_{jl} a_l \bar{a}_l a_j + \mathrm{i}\nu_j a_j. \tag{3.2}$$

In order to eliminate the explicit dependence on t_{slow} we substitute

$$a_j = b_j \exp \mathrm{i}\omega_j t_{\mathrm{slow}} \tag{3.3}$$

into (3.2) and find

$$\dot{b}_j = \frac{\mathrm{i}}{\bar{b}_j} \bar{b}_1 \bar{b}_2 \bar{b}_3 \bar{b}_4 + \mathrm{i} \sum_{l=1}^{4} \mu_{jl} b_l \bar{b}_l b_j + \mathrm{i}\xi_j b_j \tag{3.4}$$

with

$$\xi_j = \nu_j - \omega_j. \tag{3.5}$$

The set (3.4) is derivable from the Hamiltonian

$$H = 2\,\mathrm{Re}(b_1 b_2 b_3 b_4) + \tfrac{1}{2} \sum_{j,l=1}^{4}\sum \mu_{jl} b_l \bar{b}_l b_j \bar{b}_j + \sum_{j=1}^{4} \xi_j b_j \bar{b}_j \tag{3.6}$$

provided the coupling coefficients μ_{jl} are symmetric

$$\mu_{jl} = \mu_{lj}. \tag{3.7}$$

The Hamiltonian is invariant, and together with the Manley-Rowe relations, of the form

$$b_j \bar{b}_j = b_l \bar{b}_l + C_{jl} \tag{3.8}$$

one finds that (3.4) is integrable, as was already shown before (Turner 1980, Verheest 1982a).

As a possible use of the present Hamiltonian formulation one could think of the interaction between several wave quadruplets having one or more waves in common. The corresponding multiple wave-triplet interaction has already been tackled in some detail (Meiss 1979, Falk 1982, Menyuk *et al* 1983).

4. Second- and third-harmonic generation

The usual three-wave interactions obey the selection rules

$$\boldsymbol{k}_3 = \boldsymbol{k}_1 + \boldsymbol{k}_2 \qquad \omega_3 = \omega_1 + \omega_2 + \delta'. \tag{4.1}$$

A special case hereof is the second-harmonic generation (SHG), where

$$\boldsymbol{k}_2 = 2\boldsymbol{k}_1 \qquad \omega_2 = 2\omega_1 + \Delta. \tag{4.2}$$

The simultaneous generation of the second and third harmonics (S+THG) is is a very special case of two coupled wave triplets, one in normal form (4.1) and one in degenerate form (4.2). This combination has as selection rules

$$\begin{aligned} &\boldsymbol{k}_3 = 3\boldsymbol{k}_1 && \boldsymbol{k}_2 = 2\boldsymbol{k}_1 \\ &\omega_3 = 3\omega_1 + \delta && \omega_2 = 2\omega_1 + \Delta \end{aligned} \tag{4.3}$$

and was pointed out some years ago (Verheest 1976), but never thoroughly investigated. It is worth pursuing this approach a little further, in view of possible astrophysical applications. When one looks at the linear power spectrum of stellar oscillations in certain types of stars, such as ZZ-Ceti and Ap stars, one finds that most of the energy is concentrated in the fundamental and its second and third harmonics (see, for example, several papers presented at the *Joint Discussion on Solar and Stellar Non-Radial Oscillations, XIX General Assembly of the IAU, 1985*). These waves will non-linearly interact, with selection rules (4.3).

The equations governing the slow changes in the amplitudes are different from either the usual three-wave or the SHG cases and turn out to be of the form

$$\begin{aligned} &\dot{a}_1 = \mathrm{i}\lambda \bar{a}_2 a_3 \exp \mathrm{i}\delta t + \mathrm{i}\zeta \bar{a}_1 a_2 \exp \mathrm{i}\Delta t \\ &\dot{a}_2 = \mathrm{i}\mu \bar{a}_1 a_3 \exp \mathrm{i}\delta t + \mathrm{i}\xi a_1^2 \exp(-\mathrm{i}\Delta t) \\ &\dot{a}_3 = \mathrm{i}\nu a_1 a_2 \exp(-\mathrm{i}\delta t). \end{aligned} \tag{4.4}$$

A proper scaling can make the first set of coupling coefficients (λ, μ, ν) equal, but not the second (ζ, ξ) or *vice versa*!

There are several different possibilities to get rid of the explicit time dependence in (4.4). We put

$$\begin{aligned} &a_1 = b_1 \\ &a_2 = b_2 \exp(-\mathrm{i}\Delta t) \\ &a_3 = b_3 \exp[-\mathrm{i}(\delta + \Delta)t] \end{aligned} \tag{4.5}$$

and find instead of (4.4):

$$\begin{aligned} &\dot{b}_1 = \mathrm{i}\lambda \bar{b}_2 b_3 + \mathrm{i}\zeta \bar{b}_1 b_2 \\ &\dot{b}_2 = \mathrm{i}\mu \bar{b}_1 b_3 + \mathrm{i}\xi b_1^2 + \mathrm{i}\Delta b_2 \\ &\dot{b}_3 = \mathrm{i}\nu b_1 b_2 + \mathrm{i}(\delta + \Delta) b_3. \end{aligned} \tag{4.6}$$

This system is Hamiltonian when

$$\lambda = \mu = \nu \qquad \zeta = 2\xi. \tag{4.7}$$

As pointed out already, either one of these conditions can be fulfilled trivially by a suitable rescaling of the amplitudes, but the other cannot. The Hamiltonian is then

$$H = 2\lambda \ \mathrm{Re}(\bar{b}_1 \bar{b}_2 b_3) + 2\xi \ \mathrm{Re}(\bar{b}_1^2 b_2) + \Delta b_2 \bar{b}_2 + (\delta + \Delta) b_3 \bar{b}_3 \tag{4.8}$$

and invariant. Another invariant is the wave energy:

$$E = b_1 \bar{b}_1 + 2 b_2 \bar{b}_2 + 3 b_3 \bar{b}_3. \tag{4.9}$$

The factors two and three appear because of the original choice (2.3) of the action-angle variables. Such factors can be scaled out of E, but with corresponding changes in the coupling coefficients and thus in the Hamiltonian. As H and E are the only invariants at hand for the time being, the system is probably not integrable, as integrability requires a third independent invariant in involution with the other two. There are two indications pointing towards non-integrability, although they cannot constitute a complete proof of any sort.

First of all, amplitude equations such as (4.4) or (4.6) have been obtained by some sort of averaging over the fast time scale, keeping the selection rules in mind (here (4.3)). If one looks for polynomial combinations of factors such as $a_j(t_{\rm slow}) \exp \mathrm{i}(\boldsymbol{k}_j \cdot \boldsymbol{x} - \omega_j t_{\rm fast})$ (or $b_j(t_{\rm slow}) \exp \mathrm{i}(\boldsymbol{k}_j \cdot \boldsymbol{x} - \omega_j t_{\rm fast})$) which do not change on the fast time scale, then one finds $b_1\bar{b}_1$, $b_2\bar{b}_2$, $b_3\bar{b}_3$, $\mathrm{Re}(\bar{b}_1\bar{b}_2 b_3)$, $\mathrm{Re}(\bar{b}_1^2 b_2)$, $\mathrm{Re}(b_1^3\bar{b}_3)$, $\mathrm{Re}(b_1\bar{b}_2^2 b_3)$, $\mathrm{Re}(b_2^3\bar{b}_3^2)$ and combinations thereof. It is easily seen that E uses the first three of these forms and H the next two ones. It does not seem possible to combine $\mathrm{Re}(b_1^3\bar{b}_3)$, $\mathrm{Re}(b_1\bar{b}_2^2 b_3)$ and $\mathrm{Re}(b_2^3\bar{b}_3^2)$ into a third invariant, because of the different number of factors b_j in these expressions. This becomes more evident when one contrasts the case of S+THG with the four-wave interactions in two triplets with two common waves, the so-called sum and difference frequency generation, which was proved integrable by Romeiras (1983). The selection rules are

$$\begin{aligned} \boldsymbol{k}_3 &= \boldsymbol{k}_1 + \boldsymbol{k}_2 \qquad & \boldsymbol{k}_4 &= \boldsymbol{k}_1 - \boldsymbol{k}_2 \\ \omega_3 &= \omega_1 + \omega_2 \qquad & \omega_4 &= \omega_1 - \omega_2 \end{aligned} \tag{4.10}$$

and the corresponding (Hamiltonian) amplitude equations, in our complex notation:

$$\begin{aligned} \dot{b}_1 &= \mathrm{i}\lambda\bar{b}_2 b_3 + \mathrm{i}\mu b_2 b_4 \\ \dot{b}_2 &= \mathrm{i}\lambda\bar{b}_1 b_3 + \mathrm{i}\mu b_1\bar{b}_4 \\ \dot{b}_3 &= \mathrm{i}\lambda b_1 b_2 \\ \dot{b}_4 &= \mathrm{i}\mu b_1\bar{b}_2. \end{aligned} \tag{4.11}$$

If one were to look here for simple combinations which do not change on the fast time scale, one would get $b_1\bar{b}_1$, $b_2\bar{b}_2$, $b_3\bar{b}_3$, $b_4\bar{b}_4$, $\mathrm{Re}(\bar{b}_1\bar{b}_2 b_3)$, $\mathrm{Re}(\bar{b}_1 b_2 b_4)$, $\mathrm{Re}(b_1^2\bar{b}_3\bar{b}_4)$ and $\mathrm{Re}(b_2^2\bar{b}_3 b_4)$. These give precisely the four independent invariants:

$$\begin{aligned} &E = b_1\bar{b}_1 + b_3\bar{b}_3 + b_4\bar{b}_4 \qquad F = b_2\bar{b}_2 + b_3\bar{b}_3 - b_4\bar{b}_4 \\ &H = \lambda\ \mathrm{Re}(\bar{b}_1\bar{b}_2 b_3) + \mu\ \mathrm{Re}(\bar{b}_1 b_2 b_4) \\ &I = 8\lambda\mu[(\lambda^2+\mu^2)\ \mathrm{Re}(b_1^2\bar{b}_3\bar{b}_4) + (\lambda^2-\mu^2)\ \mathrm{Re}(b_2^2\bar{b}_3 b_4)] - 16\lambda^2\mu^2(b_3\bar{b}_3 b_4\bar{b}_4) \\ &\qquad -[(\lambda^2+\mu^2)b_1\bar{b}_1 - (\lambda^2-\mu^2)b_2\bar{b}_2]^2. \end{aligned} \tag{4.12}$$

A second indication is given by some numerical work of Ford and Lunsford (1970) concerning the interactions of a fundamental with some of its harmonics, although not dealing explicitly with the case of S+THG.

5. Elliptically polarised third-harmonic generation

The usual non-linear wave-wave interaction formalism is written for waves, such as simple transverse or longitudinal waves, which can each be characterised by one scalar amplitude. However, in non-linear media such as plasmas, the propagation of linearly polarised waves will be the exception rather than the rule, since the non-linearities will change the polarisation characteristics, either by rotating the plane of polarisation or by inducing a perpendicular component such that the polarisation becomes elliptical.

A more general treatment of wave interactions thus has to allow for both an elliptical polarisation of the waves and an anisotropy of the plasma or the non-linear medium. This renders the description at once much more involved, and not only quantitatively. Vector ampliudes are now needed, entailing that the wave coupling is expressed through tensors rather than scalar coefficients.

In order to get some feeling for these complications, one will want to treat first the simpler cases of SHG or of THG. In an isotropic noiseless medium no SHG is possible, hence one tackles the THG of an elliptically polarised wave in an (an)isotropic plasma, as this includes several known limiting cases.

The linearised solution is a superposition of a fundamental with amplitude $\boldsymbol{a}$ and a third harmonic with amplitude $\boldsymbol{b}$:

$$\boldsymbol{u}_{\text{lin}} = \boldsymbol{a}(t_{\text{slow}}) \exp \mathrm{i}(kz - \omega t_{\text{fast}}) + \boldsymbol{b}(t_{\text{slow}}) \exp 3\mathrm{i}(kz - \omega t_{\text{fast}}) + \text{CC}. \tag{5.1}$$

For simplicity, perfect matching has been assumed in a noiseless medium. In view of what was developed earlier, it is fairly straightforward to include a frequency mismatch or some outside noise.

The general starting point for non-linear elliptically or more generally polarised THG is then (Verheest 1982b)

$$\begin{aligned} \dot{\boldsymbol{a}} &= \mathrm{i}\mathbf{A} \vdots \boldsymbol{b}\bar{\boldsymbol{a}}\bar{\boldsymbol{a}} + \mathrm{i}\mathbf{K} \vdots \boldsymbol{a}\bar{\boldsymbol{a}}\boldsymbol{a} + \mathrm{i}\mathbf{L} \vdots \boldsymbol{b}\bar{\boldsymbol{b}}\boldsymbol{a} \\ \dot{\boldsymbol{b}} &= \mathrm{i}\mathbf{B} \vdots \boldsymbol{a}\boldsymbol{a}\boldsymbol{a} + \mathrm{i}\mathbf{M} \vdots \boldsymbol{a}\bar{\boldsymbol{a}}\boldsymbol{b} + \mathrm{i}\mathbf{N} \vdots \boldsymbol{b}\bar{\boldsymbol{b}}\boldsymbol{b}. \end{aligned} \tag{5.2}$$

The coupling tensors $\mathbf{A}$ and $\mathbf{B}$ refer to the resonant part of the interaction, whereas the others characterise the non-resonant or self-interaction or Kerr terms.

One finds that (5.2) is Hamiltonian provided

$$A_{klmn} = 3B_{nmlk} \tag{5.3}$$

and

$$\begin{aligned} K_{klmn} &= K_{mlkn} = K_{knml} = K_{lknm} = K_{nmlk} \\ L_{klmn} &= L_{lknm} = M_{mnkl} = M_{nmlk} \\ N_{klmn} &= N_{mlkn} = N_{knml} = N_{lknm} = N_{nmlk}. \end{aligned} \tag{5.4}$$

These last conditions also ensure that the Kerr terms do not change the wave energies, because

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t_{\text{slow}}}(\|\boldsymbol{a}\|^2) &\equiv \frac{\mathrm{d}}{\mathrm{d}t_{\text{slow}}}(\boldsymbol{a}\cdot\bar{\boldsymbol{a}}) = 3\mathrm{i}\mathbf{B} \vdots \bar{\boldsymbol{a}}\bar{\boldsymbol{a}}\bar{\boldsymbol{a}}\boldsymbol{b} - 3\mathrm{i}\mathbf{B} \vdots \boldsymbol{a}\boldsymbol{a}\boldsymbol{a}\bar{\boldsymbol{b}} \\ \frac{\mathrm{d}}{\mathrm{d}t_{\text{slow}}}(\|\boldsymbol{b}\|^2) &\equiv \frac{\mathrm{d}}{\mathrm{d}t_{\text{slow}}}(\boldsymbol{b}\cdot\bar{\boldsymbol{b}}) = \mathrm{i}\mathbf{B} \vdots \boldsymbol{a}\boldsymbol{a}\boldsymbol{a}\bar{\boldsymbol{b}} - \mathrm{i}\mathbf{B} \vdots \bar{\boldsymbol{a}}\bar{\boldsymbol{a}}\bar{\boldsymbol{a}}\boldsymbol{b}. \end{aligned} \tag{5.5}$$

This requirement is necessary, otherwise the method of non-linear wave coupling becomes mathematically ill-defined (Verheest 1982c). Two immediate invariants are the Hamiltonian:

$$H = 2\mathrm{Re}(\mathbf{B} \vdots \boldsymbol{a}\boldsymbol{a}\boldsymbol{a}\bar{\boldsymbol{b}}) + \tfrac{1}{2}\mathbf{K} \vdots \boldsymbol{a}\bar{\boldsymbol{a}}\boldsymbol{a}\bar{\boldsymbol{a}} + \mathbf{M} \vdots \boldsymbol{a}\bar{\boldsymbol{a}}\boldsymbol{b}\bar{\boldsymbol{b}} + \tfrac{1}{2}\mathbf{N} \vdots \boldsymbol{b}\bar{\boldsymbol{b}}\boldsymbol{b}\bar{\boldsymbol{b}} \tag{5.6}$$

and the total wave energy:

$$E = \|\boldsymbol{a}\|^2 + 3\|\boldsymbol{b}\|^2. \tag{5.7}$$

Even in the case of strict elliptical polarisation, when both a_z and b_z vanish, four independent invariants are needed and only two found! For the special case of an isotropic medium, all coupling tensors are of the form

$$T_{klmn} = \tau\delta_{kl}\delta_{mn} + \tau'\delta_{km}\delta_{ln} + \tau''\delta_{kn}\delta_{lm} \tag{5.8}$$

and (5.2) becomes, after a suitable rescaling which normalises the resonant coupling coefficients,

$$\begin{aligned}\dot{\boldsymbol{a}} &= \mathrm{i}(2\kappa\|\boldsymbol{a}\|^2 + \mu\|\boldsymbol{b}\|^2)\boldsymbol{a} + 2\mathrm{i}(\kappa' a^2 + \bar{\boldsymbol{a}}\cdot\boldsymbol{b})\bar{\boldsymbol{a}} + \mathrm{i}(\mu''\boldsymbol{a}\cdot\bar{\boldsymbol{b}} + \bar{a}^2)\boldsymbol{b} + \mathrm{i}\mu'\boldsymbol{a}\cdot\boldsymbol{b}\bar{\boldsymbol{b}}\\ \dot{\boldsymbol{b}} &= \mathrm{i}(a^2 + \mu''\bar{\boldsymbol{a}}\cdot\boldsymbol{b})\boldsymbol{a} + \mathrm{i}\mu'\boldsymbol{a}\cdot\boldsymbol{b}\bar{\boldsymbol{a}} + \mathrm{i}(\mu\|\boldsymbol{a}\|^2 + 2\nu\|\boldsymbol{b}\|^2)\boldsymbol{b} + 2\mathrm{i}\nu' b^2\bar{\boldsymbol{b}}.\end{aligned} \tag{5.9}$$

the Hamiltonian (5.6) reduces to

$$\begin{aligned}H = 2\,\mathrm{Re}(a^2\boldsymbol{a}\cdot\bar{\boldsymbol{b}}) + \kappa\|\boldsymbol{a}\|^4 + \kappa' a^2\bar{a}^2 + \mu\|\boldsymbol{a}\|^2\|\boldsymbol{b}\|^2\\ + \mu'|\boldsymbol{a}\cdot\boldsymbol{b}|^2 + \mu''|\boldsymbol{a}\cdot\bar{\boldsymbol{b}}|^2 + \nu\|\boldsymbol{b}\|^4 + \nu' b^2\bar{b}^2\end{aligned} \tag{5.10}$$

and besides the total wave energy (5.7) there is a third, new and independent invariant:

$$\boldsymbol{a}\times\bar{\boldsymbol{a}} + \boldsymbol{b}\times\bar{\boldsymbol{b}} = \boldsymbol{c}. \tag{5.11}$$

If other invariants exist for the isotropic THG, they can be proved to be of order six or higher in the wave amplitudes.

The Hamiltonian and the total wave energy are scalar invariants. For the case of pure elliptically polarised waves, with x, y components for both waves, (5.11) becomes

$$c_z = a_x\bar{a}_y - \bar{a}_x a_y + b_x\bar{b}_y - \bar{b}_x b_y \tag{5.12}$$

and one finds only three independent invariants, where four are needed to render the system completely integrable.

When $\boldsymbol{a}$ and $\boldsymbol{b}$ also include a longitudinal component, $\boldsymbol{c}$ corresponds to three scalar invariants. In this case, however, six invariants are needed, but only five have been found so far.

The indications thus seem to point to the general non-integrability of the system describing THG of elliptically polarised waves, even in isotropic media, though the set of amplitude equations is Hamiltonian. Of course, some special cases are integrable, as was known already from other work. We will cite only two examples. First, the fundamental is linearly polarised and the third harmonic is generated out of the noise. In that case it will also be linearly polarised, with the same plane of polarisation. Second, if both waves are circularly polarised and required to remain so, then the waves do not exchange energy and the non-linear interaction influences only their phases.

Especially in this paragraph, where complex vector amplitudes occur, one sees the usefulness of being able to work with complex canonical variables, because of the extreme economy in the number of equations and conditions one has to deal with. Splitting (5.2) into its real and imaginary parts and *then* checking whether and when the system is Hamiltonian or trying to find invariants such as (5.11) is so much more cumbersome and involved. Even for the scalar cases dealt with in the preceding paragraphs, the present method of using the complex wave amplitudes as (complex) canonical variables leads one as quickly as possible as far as one can get and to whatever conclusions one can draw.

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